2nd Annual Lexington Mathematical Tournament
Individual Round

Solutions

1. Answer: $\boxed{1/2011}$

   The only even prime number is 2. Since this is among the first 2011 primes, the probability of picking it is 1/2011.

2. Answer: $\boxed{11}$

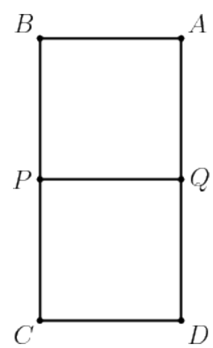
   Let H be Hansol's score and J be Julia's score; we wish to find J. We are given that

   $$J = 2H + 1$$
   $$J = H + 6.$$

   Solving, we find that $2H + 1 = H + 6 \Rightarrow H = 5 \Rightarrow J = 11$.

3. Answer: $\boxed{\sqrt{5}}$

   In naming the square, the vertices are stated *in order*, so the positions of A, B, C, D are essentially fixed as shown below.

   We can see that $CD = 1$, $BC = 1 + 1 = 2$, and thus by the Pythagorean theorem, $BD = \sqrt{1^2 + 2^2} = \sqrt{5}$.

4. Answer: $\boxed{0}$

   For each positive integer n, n and $-n$ are the two integers of distance n away from 0. Thus, including 0, there are $2n + 1$ integers that are up to distance n from 0. We want to find the sum of the 2011 integers closest to 0, so $2n + 1 = 2011 \Rightarrow n = 1005$. The sum we are looking for is thus

   $$-1005 + (-1004) + (-1003) + \cdots + (-1) + 0 + 1 + \cdots + 1004 + 1005.$$

   We can pair up the terms as

   $$\begin{aligned}(-1005 + 1005) + (-1004 + 1004) + \cdots + (-2 + 2) + (-1 + 1) + 0 &= 0 + 0 + \cdots + 0 + 0\\ &= 0.\end{aligned}$$

5. Answer: $\boxed{252}$

   From the information given, each screw costs 0.2 cents, so we can certainly buy $50/0.5 = 250$ screws and pay 50 cents. At this point, we add individual screws until the price rounds up to 51 cents. When we add 2 more screws, we have a price of 50.4 which rounds to 50, but adding 3 screws gets us 50.6 which rounds to 51, so we can have at most $250 + 2 = 252$ screws.

6. Answer: $\boxed{4}$

   For $n = 4$,
   $$a_4 + a_1 = a_2 + a_3 \Rightarrow a_4 = 1 + 2 - 1 = 2.$$
   We can compute $a_5 = 2 + 2 - 1 = 3$, $a_6 = 2 + 3 - 2 = 3$, $a_7 = 3 + 3 - 2 = 4$ similarly.

   Note: It can be shown that $a_{2n-1} = a_{2n} = n$ for all positive integers n.

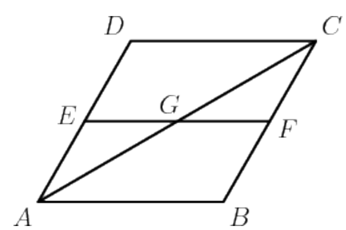
7. Answer: $\boxed{5}$

   Let h_c be the length of the altitude to $\overline{AB}$ and let h_a be the length of the altitude to $\overline{BC}$. If A is the area of the triangle, then
   $$A = \frac{1}{2}(AB)h_c = \frac{1}{2}(BC)h_a \Rightarrow (AB)h_c = (BC)h_a.$$
   We know $AB = 8$, $BC = 10$, and $h_a = 4$, so $8h_c = 10(4) \Rightarrow h_c$.

8. Answer: $\boxed{270}$

   We need to pick two of the five souls that will be the ones that came in through the same entrance, which can be done in $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$ ways. Then, for the other three souls, there are 3 entrances that each of them could have gone through (every entrance except for the one you came in through), contributing a factor of $3^3 = 27$. Overall, there are $10 \times 27 = 270$ ways exactly two souls could have come in through the same entrance as you.

9. Answer: $\boxed{1/3}$

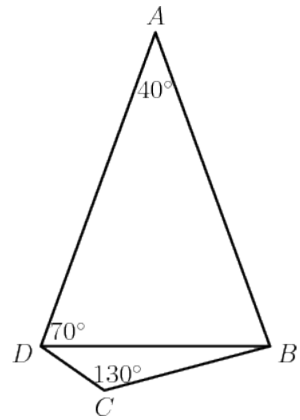


   Consider parallelogram $AEFB$ and suppose the area is equal to U, $EF = b$, and the height to $\overline{EF}$ is h. Thus, $U = bh$. Since $ABCD$ is a rhombus, G bisects $\overline{EF}$ and so triangle AEG has the same height h and half the base of parallelogram $AEFB$. Therefore, the area of AEG is equal to $\frac{1}{2}h(b/2) = U/4$. Since parallelogram $AEFB$ is made up of triangle AEG and quadrilateral $AGFB$, the area of $AGFB$ is $3U/4$. The desired ratio is thus $1/3$ or $1:3$.

10. Answer: $\boxed{54}$

    For the number to be divisible by 9, the sum of its digits must be divisible by 9. The minimum possible value for the sum is $7 \times 7 = 49$, and the maximum possible value is $7 \times 8 = 56$. The only multiple of 9 in between the two is 54.

11. Answer: $\boxed{35}$



The angles in a quadrilateral adds up 360°, so

$$m\angle A + m\angle ABC + m\angle C + m\angle CDA = 360°.$$

From the angles given, we find that

$$m\angle ABC + m\angle CDA = 360° - 40° - 130° = 190°.$$

We are given that $m\angle CDA - m\angle ABC = 20°$, so adding the equations,

$$2m\angle CDA = 210° \Rightarrow m\angle CDA = 105°.$$

We wish to find $m\angle CDB = m\angle CDA - m\angle ADB = 105° - m\angle ADB$. Since $AB = AD$, triangle BAD is isosceles and

$$\angle ADB = \frac{180° - 40°}{2} = 70°.$$

Thus, $\angle CDB = 105° - 70° = 35°$.

12. Answer: $\boxed{6}$

Regardless of the outcome of any single match, a total of 2 points are awarded. To set up a match, we must pick 2 players out of 7, which can happen in $\binom{7}{2} = 21$ ways. Since each player plays every other player exactly once, there are 21 matches and a total of $2 \times 21 = 42$ points among the players. Thus, the average number of points per player is $42/7 = 6$.

13. Answer: $\boxed{53}$

Clearly, the smallest positive integer that leaves a remainder of 3 when divided by 5 is 3. To generate all positive integers, we add multiples of 5. Since we want to find values that also leave a remainder of 4 when divided by 7, we keep adding 5's until we get such a value. The first positive integer we find that has a remainder of 3 when divided by 5 and a remainder of 4 when divided by 7 is 18. To preserve the remainder when dividing by 5, we must add multiples of 5, while to preserve the remainder when dividing by 7, we must add multiples of 7. Thus, in order to preserve both, we must add multiples of $\text{lcm}(5, 7) = 35$. The smallest positive integer with both remainder conditions satisfied is 18, so the second smallest is $18 + 35 = 53$.

For a slightly more general approach, let n be a positive integer that leaves a remainder of 3 when divided by 5 and a remainder of 4 when divided by 7. Then, for some nonnegative integers a and b,

$$n = 5a + 3 = 7b + 4.$$

Considering this equation modulo 5,

$$3 \equiv 2b - 1 \Rightarrow b \equiv 2 \pmod 5,$$

so $b = 5c + 2$ for some nonnegative integer c. Substituting, $n = 7(5c + 2) + 4 = 35c + 18$. The smallest positive integer n that satisfies this is then $35(0) + 18 = 18$ and the second smallest is $35(1) + 18 = 53$.

14. Answer: $\boxed{1880}$

$$\begin{array}{cccc} \text{L} & \text{E} & \text{E} & \text{T} \\ + & \text{L} & \text{M} & \text{T} \\ \hline \text{T} & 0 & 0 & \text{L} \end{array}$$

From the thousands digit, we know that either $L = T$ or $L + 1 = T$. From the units digit, we know that $2T$ has the same units digit as L. Thus, if $L = T$, $2L$ has the same units digit as L, and it can be seen that $L = 0$, which is forbidden, so $L + 1 = T$. With this constraint, we find that $T = 9$ and $L = 8$.

$$\begin{array}{cccc} 1 & & 1 & \\ 8 & \text{E} & \text{E} & 9 \\ + & 8 & \text{M} & 9 \\ \hline 9 & 0 & 0 & 8 \end{array}$$

For the tens digit, $1 + E + M$ has a units digit of 0, and thus $1 + E + M = 10 \Rightarrow E + M = 9$. In addition, there is a carry into the hundreds digit since $1 + E + M \geq 10$. Looking at the hundreds digit, $1 + E + L = 10 \Rightarrow E = 10 - 1 - 8 = 1$. Thus $M = 9 - 1 = 8$.

$$\begin{array}{cccc} 1 & 1 & 1 & \\ 8 & 1 & 1 & 9 \\ + & 8 & 8 & 9 \\ \hline 9 & 0 & 0 & 8 \end{array}$$

Answering the question, the value of the four-digit number ELMO is 1880.

15. Answer: $\boxed{72}$

By definition,

$$\begin{aligned} 11! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \\ &= 2^1 \cdot 3^1 \cdot 2^2 \cdot 5^1 \cdot 2^1 \cdot 3^1 \cdot 7^1 \cdot 2^3 \cdot 3^2 \cdot 2^1 \cdot 5^1 \cdot 11^1 \\ &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1 \cdot 11^1. \end{aligned}$$

Thus, we have

$$\begin{aligned} 20 \cdot n^2 &| 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1 \cdot 11^1 \\ n^2 &| 2^6 \cdot 3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1. \end{aligned}$$

Since n clearly cannot contain any other prime factor, if we let $n = 2^a 3^b 5^c 7^e 11^f$, then we have

$$\begin{aligned} 2a \leq 6 &\Rightarrow \max a = 3 \\ 2b \leq 4 &\Rightarrow \max b = 2 \\ 2c \leq 1 &\Rightarrow \max c = 0 \\ 2d \leq 1 &\Rightarrow \max d = 0 \\ 2e \leq 1 &\Rightarrow \max e = 0. \end{aligned}$$

The largest value that n can be is then $2^3 \cdot 3^2 = 72$.

16. Answer: $\boxed{2}$

Let S be the common value of the sum of the numbers in each row, column, and long diagonal. Adding up the totals in each row, we get $3S$. However, since we added up the numbers in all the rows, this sum is equal to the sum of all the numbers used, so

$$3S = 11 + 12 + 13 + \cdots + 18 + 19 = 9(11 + 19)/2 = 135 \Rightarrow S = 45.$$

Looking at the long diagonal containing 18 and 15, since the sum of the three numbers is 45, the last number is $45 - 18 - 15 = 12$.

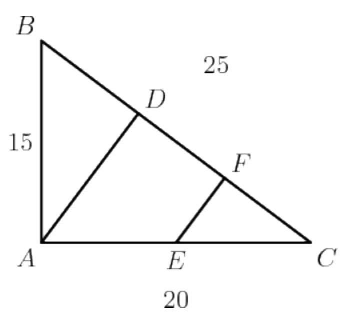
		18
	15	
12		

Now, suppose the number in the bottom right is a. The number in row 2, column 3 must then be $45 - 18 - a = 27 - a$. Since we are using integers from 11 to 19 inclusive, $11 \leq 27 - a \leq 19 \Rightarrow 8 \leq a \leq 16$. Similarly, the number in row 3, column 2 must be $45 - 12 - a = 33 - a$. By the same restrictions, $11 \leq 33 - a \leq 19 \Rightarrow 14 \leq a \leq 21$, so $14 \leq a \leq 16$ when we combine the two inequalities. Since 15 has already been used, we can only possibly have $a = 14$ or $a = 16$. Testing both of these finds that we have 2 possible magic squares, shown below.

16	11	18
17	15	13
12	19	14

14	13	18
19	15	11
12	17	16

17. Answer: $\boxed{72}$



Using the Pythagorean theorem, we have $15^2 + 20^2 = (BC)^2 \Rightarrow BC = 25$. Since AD is the height from BC, we know that

$$\begin{aligned} \text{Area} = \frac{AD \cdot BC}{2} &= \frac{AB \cdot AC}{2} \\ \frac{25 \cdot AD}{2} &= \frac{15 \cdot 20}{2} \\ AD &= 12. \end{aligned}$$

Again by the Pythagorean theorem, we have $CD = \sqrt{20^2 - 12^2} = 16$. The area of triangle ACD is thus

$$\begin{aligned}\frac{AD \cdot DC}{2} &= \frac{12 \cdot 16}{2} \\ &= 96.\end{aligned}$$

Since $\triangle ECF$ and $\triangle ACD$ share $\angle C$ and a right angle at F and D, they are similar and oriented the same way. Because E is the midpoint of $\overline{AC}$, the dimensions of ECF are half of the dimensions of ACD, so the area of triangle ECF is $\left(\frac{1}{2}\right)^2 \cdot 96 = 24$. The area of $ADFE$ is then $[ACD] - [ECF] = 96 - 24 = 72$, where the square brackets denotes area.

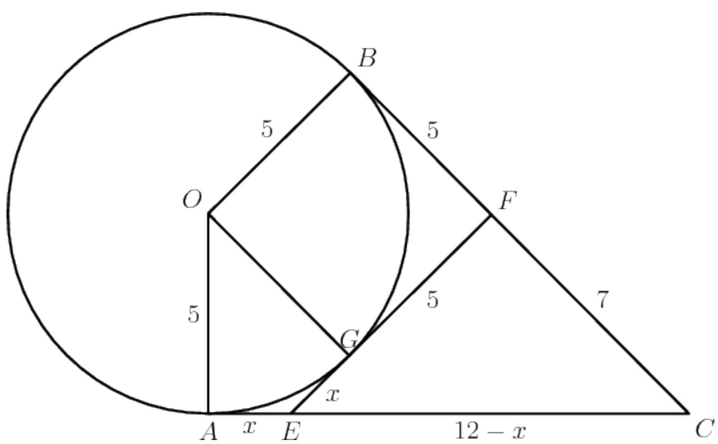
18. Answer: $\boxed{49}$

Since $2x + 1$ and $2y + 1$ are both squares, they must be an odd square under $2 \cdot 15 + 1 = 31$. Thus, $2x + 1$ and $2y + 1$ can equal 9 or 25, corresponding to x or y values of 4 or 12. (if one of them was equal to 1, then one of x or y would be 0, which is not positive). Since x and y are distinct, they must be equal to 4 and 12 in some order. In addition, when this is the case, $xy + 1 = 49$ is a perfect square, so the value we seek is 49.

19. Answer: $\boxed{282}$

We first consider the unit digits of the three factors and the product. Since the product has a units digit of 8, none of the factors can have a 0 in the units digit spot, so the unit digits must be 2, 4, 6 or 4, 6, 8. Multiplying this out, $2 \times 4 \times 6 = 48$ has a units digit of 8 and $4 \times 6 \times 8 = 192$ has a units digit of 2, so only $2, 4, 6$ can be the units digits of the three factors. We then note that the factors are all relatively close to each other, so the product should be close to a perfect cube. We see that $100^3 = 1000000$ and $90^3 = 72900$, so the only possible factors are 92, 94, and 96. Multiplying these out we see that the product is equal to 830208, so they satisfy the requirements. The sum of these numbers is 282.

20. Answer: $\boxed{420/17}$



We draw radius $\overline{OG}$ such that G is the point of tangency with circle O with line segment $\overline{EF}$. We see that $OBFG$ is a square, as it has 90 degree angles and $OB = OG$. Since $OG = 5$,

$$BF = 5 \Rightarrow FC = 12 - 5 = 7.$$

Now, let $AE = x$, so $EG = x$ as well since they are tangents to the circle from the same point. We also know that $FG = 5$ since $OB = 5$. Thus, $EC = 12 - x$ and $EF = 5 + x$. By the Pythagorean Theorem,

$$\begin{aligned} EF^2 + FC^2 &= EC^2 \\ (5+x)^2 + 7^2 &= (12-x)^2 \\ x^2 + 10x + 25 + 49 &= x^2 - 24x + 144 \\ x &= \frac{35}{17}. \end{aligned}$$

EF is thus equal to $\frac{35}{17} + 5 = \frac{120}{17}$. Since triangle EFC is right, its area is $\frac{120}{17} \cdot 7 \cdot \frac{1}{2} = \frac{420}{17}$.